Lie algebras, Lie groups.

Logistics No HW this week

Dfn . Lie group

. Ex $(\mathbb{R}, +)$

$(GL_n\mathbb{R}, \circ)$

mult notation: good to think abt subgps of GL_n

. Action

Ex G acts on itself in 2 ways : $L_g h = gh$, $R_g h = hg^{-1}$ } commute

Ex: If V complete, Φ gives an action of $\mathbb{R}$ on M.

Observation $\mathbb{R}$-actions on M $\longleftrightarrow$ complete vector fields.

"generator" of the action

⌜ If $\Phi: \mathbb{R}\times M \to M$ is an action, then $V := \frac{\partial \Phi}{\partial t}\Big|_{t=0} \in \mathrm{Vect}(M)$ and Φ = flow of V ⌟

When is V complete?

① M cpct ⌜ ⌟

② V left-inv't ⌜ ⌟

Dfn Lie algebra $\mathfrak{g}$ of G : left-inv't VF's $\longleftrightarrow T_e\mathfrak{g}$

Thm $\mathfrak{g} \longleftrightarrow$ 1-parameter subgroups of G

We'll say $\Phi: \mathbb{R}\times G \to G$ is a right action if $\Phi(t,g) = R_{F(t)}\, g$ for a 1PS F.

i.e. X left-inv't $\iff$ Φ is a right action

[① left-invariance $\Rightarrow$ (X le inv't $\Leftrightarrow$ flow of X le-inv't.)

② ($\Leftarrow$) Right action commutes w/ left action

③ ($\Rightarrow$) Let $F(t) = \Phi(-t, e)$. Need to show A) $F: \mathbb{R} \to G$ is a homo and B) $\Phi(t, g) = g\, F(t)^{-1}$

By ①, Φ left-inv't, so $h\,\Phi(t,g) = \Phi(t, gh)$

So

A) $F(s)F(t) = F(s)\Phi(-t, e) = \Phi(-t, \Phi(-s, e)) = \Phi(-s-t, e) = F(s+t)$

B) $\Phi(t, g) = g\,\Phi(t, e)$

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Defn For $X \in \mathfrak{g}$, $e^{tX} := \Phi(t, e)$. Exponential $\mathfrak{g} \to G$.

IFT $\Rightarrow$ local diffeo

Ex (Boothby) $\mathrm{Lie}(GL_n) = \mathrm{Mat}(n)$; $e^{tX} = \frac{I}{0!} + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots$

recall

Lie derivative:

If X, Y vect fields, Φ flow of X, then $((\Phi_t)_* Y)_p = d\Phi_t Y_{\Phi_t^{-1}(p)}$

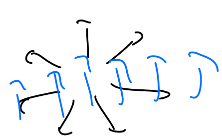
Defn $\mathcal{L}_X Y\big|_p = -\frac{d}{dt}\Big|_{t=0} ((\Phi_t)_* Y)_p$

↑ will see why.

compare $Xf = \lim \frac{f\circ\Phi_t - f}{t} = \frac{d}{dt}\Big|_{t=0} \Phi_t^* f$

eg $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

$Y = \frac{\partial}{\partial y}$



$\Phi(p,t) = e^t p$

$d\Phi_t Y = \frac{\partial}{\partial y}(e^t p) = e^t \frac{\partial}{\partial y} = e^t Y$

$[X,Y] = -\frac{\partial}{\partial y}$

In this ex, $\mathcal{L}_X Y = -\mathcal{L}_Y X$

There's a wonderful trick to compute $\mathcal{L}_X$'s without studying Φ_X.

Uses alternative char of Vect(S)

$C^\infty(S) :=$ smooth fns $S \to \mathbb{R}$ is an $\mathbb{R}$-algebra (vs/$\mathbb{R}$, mult)

$\varphi : S \to \hat{S}$ realizes $C^\infty(S)$ as a $C^\infty(\hat{S})$-module $f \cdot g = (f \circ \varphi) g$

We've seen $V \in$ Vect(S) gives way to differentiate fns Vf (lin)

<u>Defn</u> If R is an $\mathbb{R}$-algebra, M a (R's) R-module, a derivation from R to M is an $\mathbb{R}$-linear map $D : R \to M$ satisfying

$$D(fg) = D(f) g + f D(g)$$

Then Leibniz $\Rightarrow$

V vector field along f (section of $f^* T\hat{S}$) determines a derivation $C^\infty(\hat{S}) \to C^\infty(S)$

<u>Prop</u> If $p \to S$ is a point, $\mathcal{U}$ (the image of) a chart at p,

① every derivation $D : C^\infty(\mathcal{U}) \to \mathbb{R}_p$ comes from a $V_p \in T_p S$

② every derivation $D : C^\infty(S) \to C^\infty(S)$ - - - $V \in$ Vect(S)

$\vdash$ ① set $V^i = D(x^i)$. Taylor $\Rightarrow$

any $f = f(0) + \sum_i x^i g_i(x)$, $g_i(p) = \frac{\partial f}{\partial x^i}$

$$Df = D(f(0)) + \sum_i x^i_{(p)} D(g_i) + \sum_i V^i g_i(p)$$

$D(1) = 2D(1) = 0$

$\mathbb{R}$-lin $\to 0$

② bump fn $V^i = D(\eta x^i)$ $\quad \eta = $ $\quad \hat{\eta} =$

$$D(\hat{\eta})_p = D(\eta \hat{\eta})_p = D(\eta)_p + D(\hat{\eta})_p$$

$\Rightarrow D(\eta)_p = 0$

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